Technical Comments

Comment on "Equilibrium Orientations of Gravity-Gradient Satellites"

THOMAS R. KANE* AND PETER W. LIKINS†
Stanford University, Stanford, Calif.

A GLANCE at Eqs. (1–3), reproduced below from Ref. 1, shows that

$$\alpha = \beta = \gamma = 0$$

is the only solution that exists when $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ are identically zero:

$$A\ddot{\alpha} + 3\Omega^2(B - C)\alpha = 0 \tag{1}$$

$$B\ddot{\beta} + 4\Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma} = 0 \tag{2}$$

$$C \quad + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta} = 0 \tag{3}$$

Consequently, the author's assertion that there exists "an entire one-parameter family of points in α, β, γ space" which represents an orientation of dynamic equilibrium is incorrect.

Apparently, this conclusion was reached by mistakenly interpreting Eq. (5) as a condition that guarantees equilibrium, whereas the correct interpretation is the converse, namely, that equilibrium guarantees the validity of Eq. (5).

Reference

[1] Michelsen, I., "Equilibrium orientations of gravity-gradient satellites," AIAA J. 1, 493 (1963).

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* Professor of Engineering Mechanics.
† Graduate Student.

Reply by Author to T. R. Kane and P. W. Likins

IRVING MICHELSON*
Illinois Institute of Technology, Chicago, Ill.

ALTHOUGH the result (5)† was obtained by direct integration of the approximate differential system (1–3), a more fundamental demonstration may be preferred by readers who are unfamiliar with analogous conclusions in celestial mechanics. The fact that an integral of energy is available leads at once to a complete map of equilibrium orientations in angular coordinate space, as shown below. A further refinement is the assignment of accurate limits to regions of possible equilibrium orientation, by recourse to the full nonlinear dynamical formulation. The argument that follows is motivated in part by Hill's deductions based on the use of Jacobi's integral in the restricted problem of three bodies.

The usual set of detailed libration equations (1–3) exhibits the motion as a conventional sixth-order differential system, there being three degrees of rotational freedom. A better point of departure is available, however, whenever the Lagrangian function is not an explicit function of time. This is just the case for circular orbits, and an integral of energy is written at once as

$$\dot{\alpha}\frac{\partial L}{\partial \dot{\alpha}} + \dot{\beta}\frac{\partial L}{\partial \dot{\beta}} + \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}} - L = \text{const} \tag{6}$$

where L denotes the Lagrangian function. Equation (6) is then the complete expression of the dynamical (e.g., Hamiltonian) principle, and being of third order only, is advantageous for present purposes, as compared with (1–3). It is noted that condition (4), earlier obtained by three integrations of the original system, is also of third order, conforming with (6).

By setting the angular velocities all simultaneously equal to zero, $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$, (6) becomes the defining relationship for "permanent configurations" here sought as librational equilibrium orientations. Configuration coordinates are then given by (6) as a function of total system energy. Owing to the presence of gyroscopic terms in the kinetic energy, the left member of (6) takes a form slightly different from the potential energy itself, giving finally the equation

$$(3a_3^2 - a_1^2)A + (3b_3^2 - b_1^2)B + (3c_3^2 - c_1^2)C = \text{const} \tag{7}$$

The symbols $a_1, \ldots, c_3$ are introduced as principal axis (x_1, x_2, x_3) direction cosines, determined by the ordered sequence of finite rotations α, β, γ starting from the reference axes (X_1, X_2, X_3) themselves rotating with orbital angular speed Ω; in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{8}$$

The more accurate and complete form (7) of the equations defining equilibrium orientations is both an extension of (5) and its rigorous justification. When no restriction is made on the principal moments A, B, C, (7) is a transcendental equation in all three angular variables α, β, γ. In the trivial limiting case of complete dynamical symmetry, $A = B = C$, the basic property of direction cosines reduces the left member to a constant: independence of α, β, γ then shows that all orientations are possible permanent configurations. Intermediate degrees of dynamical symmetry evidently determine equilibrium zones of lesser extent, but nevertheless infinite in number of points (i.e. orientations) represented. Generalization to elliptic orbit in the first instance entails the loss of energy integral (7), but standard canonical transformation and perturbation procedures are then applicable and well suited to orbits of moderate eccentricity. It is intuitively clear that attendant reduction of equilibrium region increases with orbit eccentricity. Total degeneracy of the equilibrium zone into discrete points and corresponding discrete equilibrium orientations has been demonstrated by J. L. Synge by taking the fictitious limit of zero moment of inertia in the earth-pointing direction $C = 0$, as indicated in Ref. 1.

For the sake of completeness, the explicit forms of kinetic and potential energies and the relevant direction cosines are tabulated as follows:

$$2T = A(\dot{\alpha}\cos\beta\cos\gamma + \dot{\beta}\sin\gamma + a_1\Omega)^2 + B(-\dot{\alpha}\cos\beta\sin\gamma + \dot{\beta}\cos\gamma + b_1\Omega)^2 + C(\dot{\alpha}\sin\beta + \dot{\gamma} + c_1\Omega)^2 \tag{9}$$

$$2U = 3\Omega^2[Aa_3^2 + Bb_3^2 + Cc_3^2] \tag{10}$$

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* Professor of Aerospace Engineering. Member AIAA.
† Equation numbers refer to and notation conforms with Ref. 1.

$$\begin{aligned} a_1 &= \cos\beta\cos\gamma & a_3 &= \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ b_1 &= -\cos\beta\sin\gamma & b_3 &= \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ c_1 &= \sin\beta & c_3 &= \cos\alpha\cos\beta \end{aligned} \quad (11)$$

Portions of the potential which are independent of angular orientation have been omitted from (10), which is seen to be a form of MacCullagh's expression for the potential of a distant attracting mass.

It is hardly necessary to emphasize that the result enunciated in Ref. 1 is correctly scrutinized by applying the more succinct principle (7) without approximation or special assumptions, as indicated previously. Readers who might still cling to the excessively differentiated system (1–3) should recognize that (4) demonstrates an invariance of the differential system with respect to displacement of origin of angular coordinates; this provides an alternative interpretation of the result (5) or (7).

The angle variables employed in the present discussion are seen to possess the same advantages of convenience secured in a different manner by Jeffreys.[2]

For more complete discussions of special integrals of energy, of permanent configurations and surfaces of zero velocity, of effects of gyroscopic terms, and of other concepts utilized in the preceding discussion, the reader should consult standard treatises such as Refs. 3 and 4.

References

[1] Michelson, I., "Equilibrium orientations of gravity-gradient satellites," AIAA J. **1**, 493 (1963).

[2] Jeffreys, H., "The moon's principal librations in rectangular coordinates," Monthly Notices Roy. Astronom. Soc. **115**, 189 (1955).

[3] Brouwer, D. and Clemence, G. M., *Methods of Celestial Mechanics* (Academic Press, Inc., New York, 1961), Chap. X.

[4] Moulton, F. R., *An Introduction to Celestial Mechanics* (Macmillan Company, New York, 1914), revised ed., Chap. VII.

Comment on "Coning Effects Caused by Separation of Spin-Stabilized Stages"

R. O. Wilke*
Naval Missile Center, Point Mugu, Calif.

In a previous note,[1] Dwork discussed a separation effect that produces instantaneous variation in coning angle upon separation of spin-stabilized stages. The effect is purely mechanical and owes its origin to simple conservation of the moment of momentum vector. However, somewhere between the desk pen and printed copy an error arises which could result in serious consequences should the equations be applied directly. A correction of the equations will also result in a clearer physical picture.

Following Dwork, we start with the magnitude of the moment of momentum vectors about the longitudinal and transverse axes prior to separation of the two bodies:

$$\left.\begin{aligned} M_L &= (I_{L_1} + I_{L_2})\omega_L \\ M_T &= (I_{T_1} + I_{T_2})\omega_T + \mu l^2 \omega_T \end{aligned}\right\} \quad \mu = \frac{m_1 m_2}{(m_1 + m_2)}$$

The term μl^2 can easily be shown to represent a contribution to transverse moment of inertia through axis transformation and should appear in the transverse equation as just shown and not in the longitudinal.

The rest of the discussion follows in order, but the following observation might be made. The coning angle just prior to separation is defined as

$$\tan\theta_{\text{sep}} = \frac{M_T}{M_L} = \frac{(I_{T_1} + I_{T_2} + \mu l^2)\omega_T}{(I_{L_1} + I_{L_2})\omega_L}$$

and after separation it becomes (for the first mass)

$$\tan\theta_1 = \frac{M_{T_1}}{M_{L_1}} = \frac{I_{T_1}}{I_{L_1}} \tan\theta_{\text{sep}} \frac{(I_{L_1} + I_{L_2})}{(I_{T_1} + I_{T_2} + \mu l^2)}$$

Therefore,

$$\tan\theta_1 = \left(\frac{I_T}{I_L}\right)_{\text{payload}} \tan\theta_{\text{sep}} \left(\frac{I_L}{I_T}\right)_{\text{combined}}$$

The transverse moment of inertia term with the transformation quantity μl^2 merely means the transverse moment of inertia about the common center of gravity. Since this quantity is usually measured for each vehicle and combination of vehicles, it might be more convenient to refer to the foregoing in terms of moment of inertia ratios before and after separation. This ratio comparison gives a better physical idea of the losses or gains to be expected instaneously even with a "perfect" separation system:

$$\frac{\tan\theta_1}{\tan\theta_{\text{sep}}} = \frac{(I_L/I_T)_{\text{combined}}}{(I_L/I_T)_{\text{payload}}}$$

Reference

[1] Dwork, M., "Coning effects caused by separation of spin-stabilized stages," AIAA J. **1**, 2639–2640 (1963).

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* Flight Mechanics Engineer, Controls Division, Laboratory Department.

Reply by Author to R. O. Wilke

M. J. Dwork*
Radio Corporation of America, Princeton, N. J.

The author wishes to thank R. O. Wilke. The comments made are correct, errors do exist in the subscripts. Moreover, the relations he derives are simpler in form to use.

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* Senior Engineer, Astro-Electronics Division; formerly with Airborne Instruments Laboratory, Deer Park, N. Y.

Comments on "Integral Approach to an Approximate Analysis of Thrust Vector Control by Secondary Injection"

Wayne Simon*
Martin Company, Denver, Colo.

It is instructive to compare the results of the present paper[1] with other work,[2,3] which also used an integral approach to the problem of thrust vector control by secondary injection. The assumptions of Ref. 2 and 3 were markedly different than those of the Ref. 1. Specificially, the mathematical model was based on one-dimensional, inviscid, isentropic

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* Associate Research Scientist, Aerospace Division. Member AIAA.